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FO = First-Order logic

Vocabulary	Relational symbols:	$\Sigma = \{R, S, T, \ldots\}$	(aka signature)
	Variables:	$x, y, \ldots, x_1, x_2, \ldots$	
	Quantifiers:	$\exists, \forall$	
	Boolean connectives:	$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$	

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Syntax

$$\phi : \quad R(x_1,\ldots,x_k) \mid \ldots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi$$

$$\exists x\, \phi \mid \forall x\, \phi \mid \ldots$$

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ϕ : $R(x_1,\ldots,x_k)$ | ... | $\phi \vee \phi$ | $\phi \wedge \phi$ | $\neg\phi$ | $\phi \rightarrow \phi$ | $\phi \leftrightarrow \phi$

$\exists x\, \phi$ | $\forall x\, \phi$ | ...

Semantics

Now a model consists of a universe U^M

+ some mappings $R \mapsto R^M \subseteq U^M \times \ldots \times U^M$

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$\quad \exists x\, \phi \;|\; \forall x\, \phi \;|\; \ldots$

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+ some mappings $R \mapsto R^M \subseteq U^M \times \ldots \times U^M$

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$M \models \phi_1 \vee \phi_2$ iff $M \models \phi_1$ or $M \models \phi_2$

...

$M \models R(x_1,\ldots,x_k)$ iff $(x_1^M,\ldots,x_k^M) \in R^M$

$M \models \exists x\, \phi$ iff $M[x:=u] \models \phi$ for *some* $u \in U^M$

$M \models \forall x\, \phi$ iff $M[x:=u] \models \phi$ for *every* $u \in U^M$

Examples

Syntax $\phi: \quad R(x_1,\ldots,x_k) \;|\; \ldots \;|\; \phi \vee \phi \;|\; \phi \wedge \phi \;|\; \neg\phi \;|\; \phi \rightarrow \phi \;|\; \phi \leftrightarrow \phi$

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"All humans are mortal. Socrates is human. So Socrates is mortal."

$$\phi(y) \;=\; (\,(\forall x\, A(x) \rightarrow B(x))\;\&\; A(y)\,) \rightarrow B(y)$$

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M : $U^M = \{\text{Socrates, Plato, Cyclop, Jupiter}\}$

$A^M = \{\text{Socrates, Plato}\}$

$B^M = \{\text{Socrates, Plato, Cyclop}\}$

$y^M = \text{Socrates}$

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"There is a node in the graph that is isolated from all other nodes."

$$\phi = \exists x\, \forall y\, \neg(x=y) \rightarrow \neg E(x,y)$$

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$\qquad E^M = \{\text{edges of a graph}\}$

Examples

Syntax $\phi : \quad R(x_1,\ldots,x_k) \mid \ldots \mid \phi \lor \phi \mid \phi \land \phi \mid \neg\phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi$

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"There's a man such that when he runs, everybody runs."

$$\phi = \exists x\; R(x) \rightarrow \forall y\, R(y)$$

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M' : $U^{M'}$ = {Ben, Han, Leia, Luke}
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Examples

- “R is a function” $\phi = \forall x\, \exists y\ R(x,y) \ \wedge\ \forall z\, R(x,z) \rightarrow y=z$

  in this case, one can use the shorthand

  “R(x)=...” for $\exists y\, R(x,y) \ \wedge\ \forall z\, R(x,z) \rightarrow z=\ldots$

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- "+ admits *zero* and *inverses*" $\phi = \exists x_0\;\; \forall y\;\; x_0{+}y = y \;\wedge\; \forall y\, \exists z\;\; y{+}z = x_0$

Exercices

- "f is continuous" $\phi = \forall x\ \forall \varepsilon\ \exists \delta\ \forall y\ \ ||x-y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

- "f is uniformly continuous" $\phi = \forall \varepsilon\ \exists \delta\ \forall x\ \forall y\ \ ||x-y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

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What is an appropriate <u>signature</u> for the above formulas?

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- "f is uniformly continuous" $\phi = \forall \varepsilon\ \exists \delta\ \forall x\ \forall y\ \ ||x-y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

What is an appropriate signature for the above formulas?

Are the formulas equivalent? Is one a consequence of another? Can you prove it?

(hint: $\exists x\ \forall y\ \alpha \rightarrow \forall y\ \exists x\ \alpha$ *assuming universe is non-empty*)

Exercices

Choose appropriate universes and signatures, and define these properties in FO:

1. "There are infinitely many Prime numbers" $\phi = \ldots$

2. "In the tree, z is the least common ancestor of x and y" $\phi(x,y,z) = \ldots$

3. "Polynomial p evaluates to y on x" (for fixed p) $\phi_p(x,y) = \ldots$

4. "The graph is strongly connected" $\phi = \ldots$

5. "In the infinite sequence of a's and b's, every a is followed by b" $\phi = \ldots$

Normal forms

Prenex [+CNF/DNF] as for QBF, i.e. $\phi = Qx_1 \ldots Qx_n \ \alpha(x_1,\ldots,x_n)$

NNF (Negation Normal Form)

$\phi : \quad \exists x\, \phi \ \mid \ \forall x\, \phi \ \mid \ \phi \vee \phi \ \mid \ \phi \wedge \phi \ \mid \ \alpha$

$\alpha : \quad R(x_1,\ldots,x_k) \ \mid \ \neg\, R(x_1,\ldots,x_k)$

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Lemma Given ϕ ($\leftrightarrow$-free), one can compute in polynomial time an *equivalent* formula ϕ^* in NNF

Proof As for propositional logic, push negations inside:

$$\neg\forall\phi \rightsquigarrow \exists\neg\phi$$
$$\neg\exists\phi \rightsquigarrow \forall\neg\phi$$
$$\neg(\phi_1 \wedge \phi_2) \rightsquigarrow \neg\phi_1 \vee \neg\phi_2$$
$$\neg(\phi_1 \vee \phi_2) \rightsquigarrow \neg\phi_1 \wedge \neg\phi_2$$

Algorithms

<u>Model-checking</u> problem

input: formula ϕ + *finite* model M
output: yes iff $M \vDash \phi$

<u>Satisfiability</u> problem

input: formula ϕ
output: yes iff $M \vDash \phi$ for *some* M

(recall: ϕ <u>valid</u> iff $\neg\phi$ is not satisfiable
ϕ, ϕ' <u>equivalent</u> iff $\phi \leftrightarrow \phi'$ is valid)

Algorithms

🎉 **PSPACE** 🎉

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💀 **UNDECIDABLE** 💀

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Algorithms — model-checking

```
Model-check(φ, M)
    if  φ = R(x1,…,xk)  then
        if  (x1^M,…,xk^M) ∈ R^M  then
            return true
        else
            return false
    else if  φ = φ1 ∨ φ2  then
        return Model-check(φ1, M)  OR
                 Model-check(φ2, M)
    else if …
    …
    else if  φ = ∃x φ’  then
        for  u ∈ U^M  do
            if  Model-check(φ’, M[x:=u])  then
                return true
        return false
    else if  φ = ∀x φ’  then
        for  u ∈ U^M  do
            if  NOT  Model-check(φ’, M[x:=u])  then
                return false
        return true
```

Algorithms — satisfiability

Theorem [Trakhtenbrot ’50] Satisfiability of FO is undecidable

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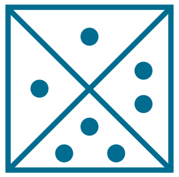
Proof by reduction from Domino (aka Tiling) problem...



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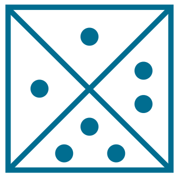


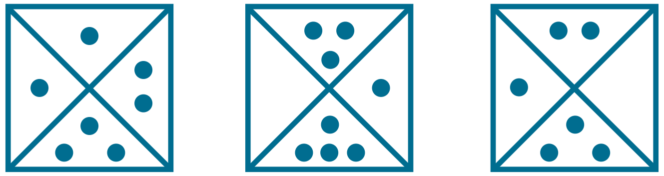
<u>Reduction</u> from P to P': Algorithm A that solves P by using an oracle that returns solutions to P'

(think of "P easier than P'")

e.g. many-one reduction: for all x P(x) iff P'(A(x))

The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

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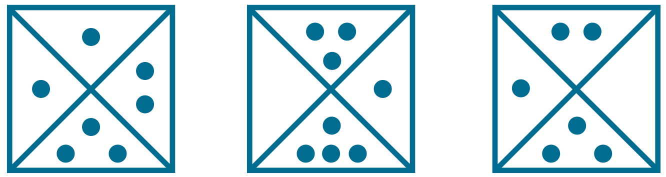
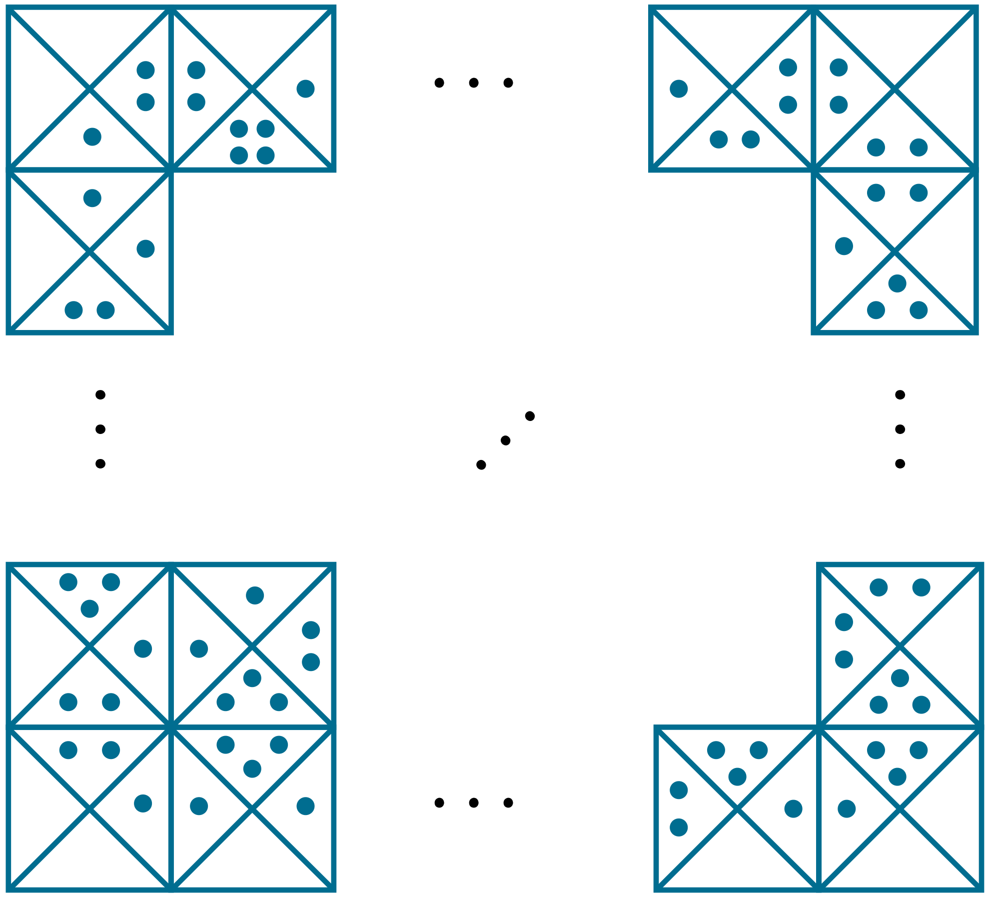
Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)



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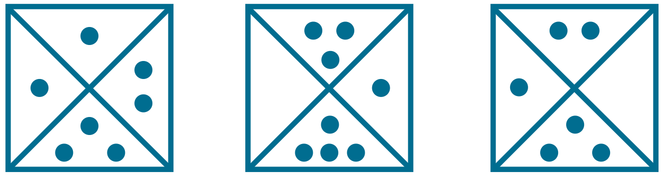

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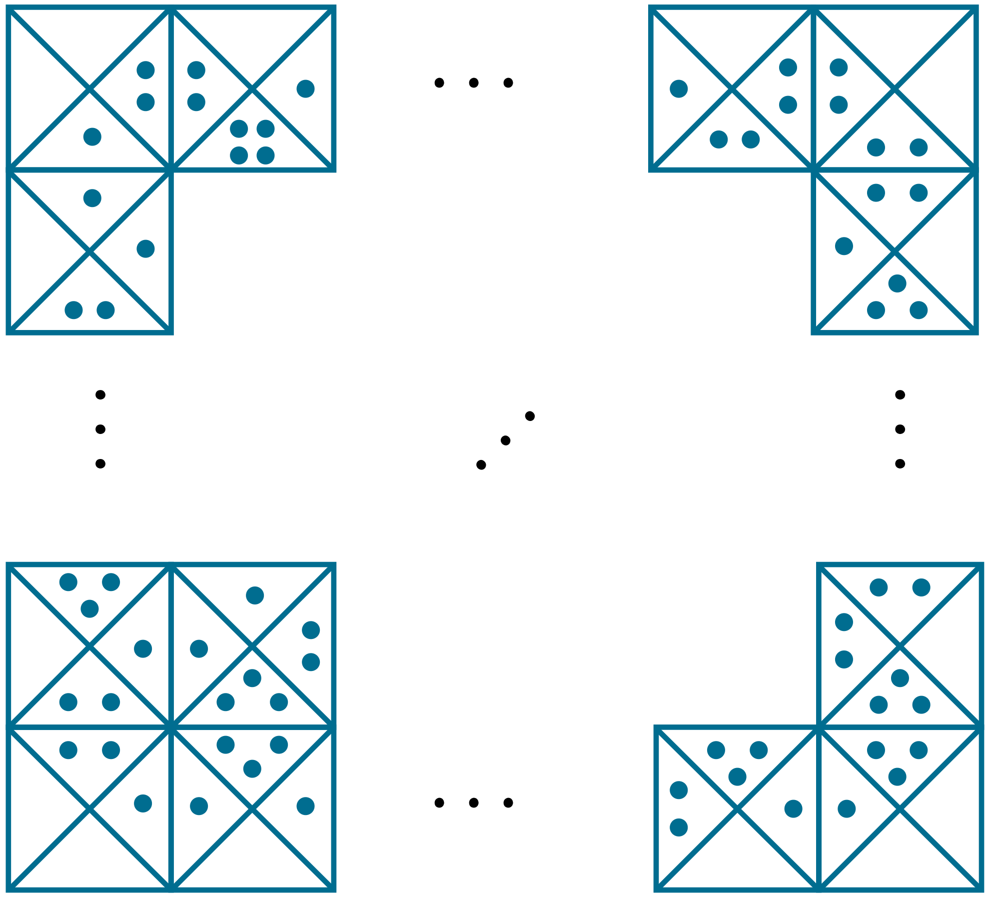
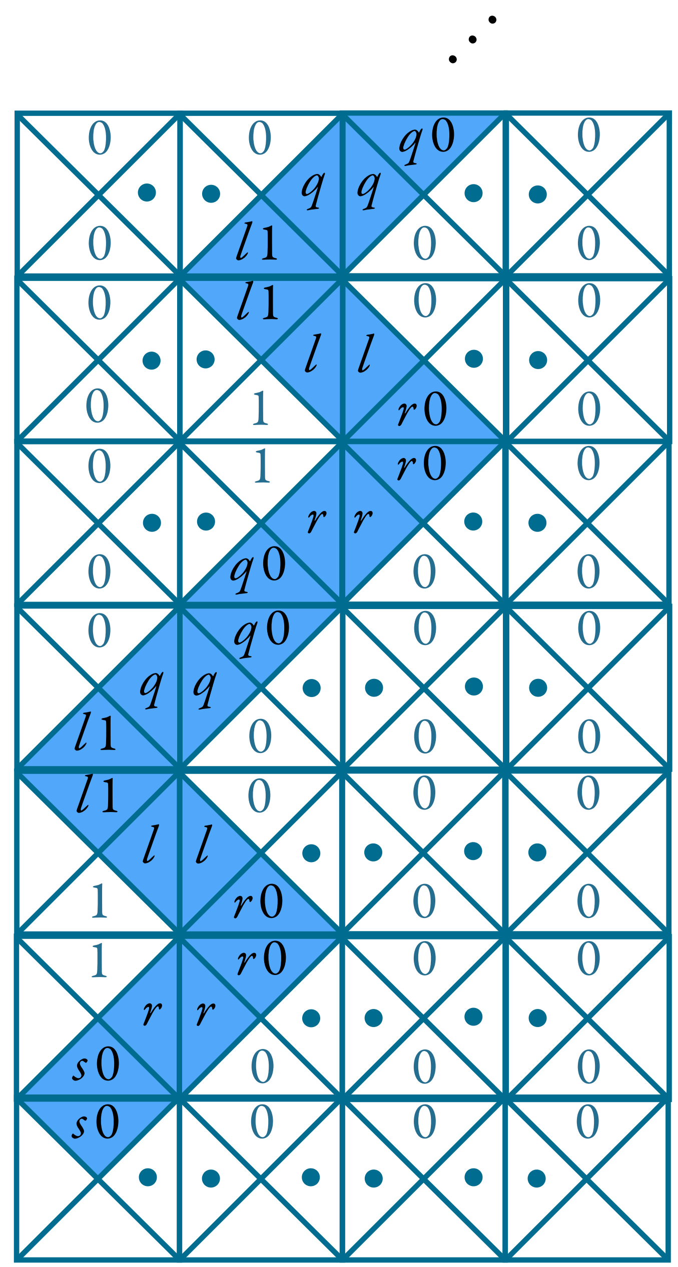


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

The (undecidable) Domino problem

Domino - Why is it undecidable?

It can encode *halting* computations of Turing machines:



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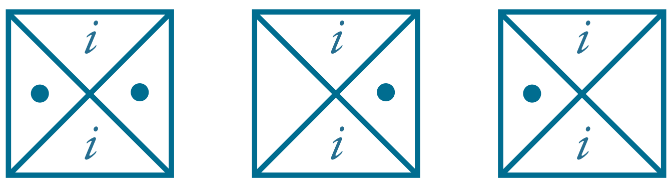

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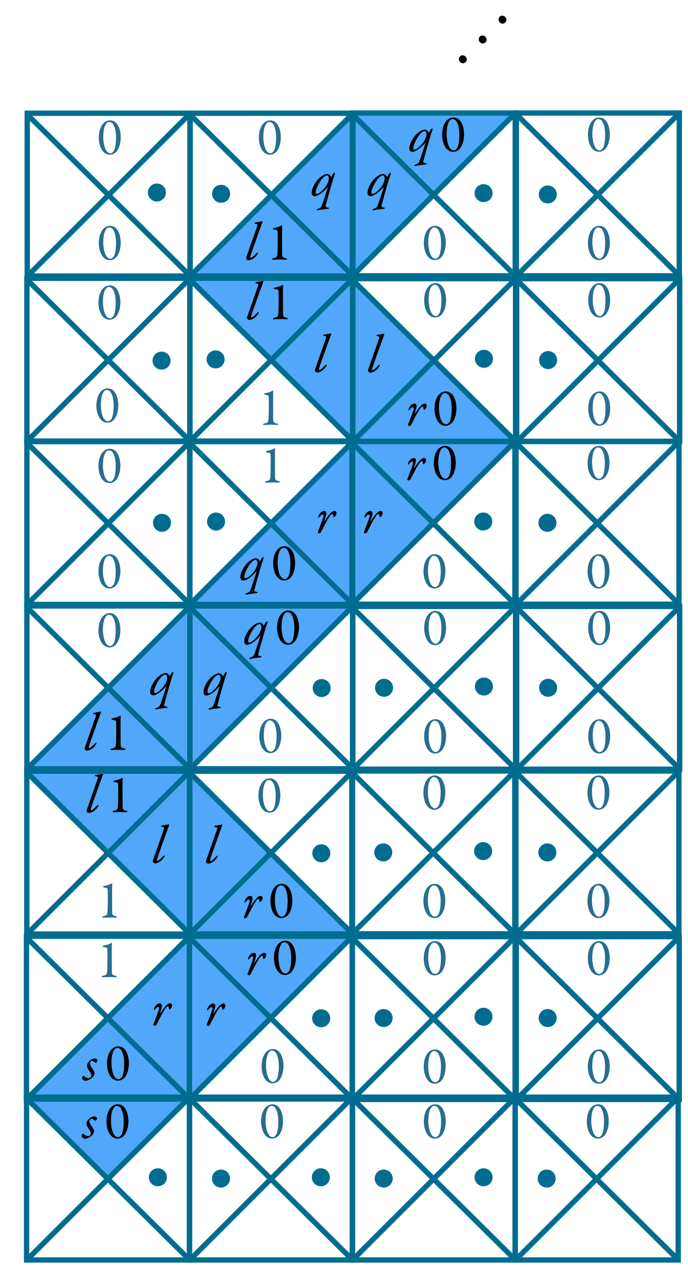

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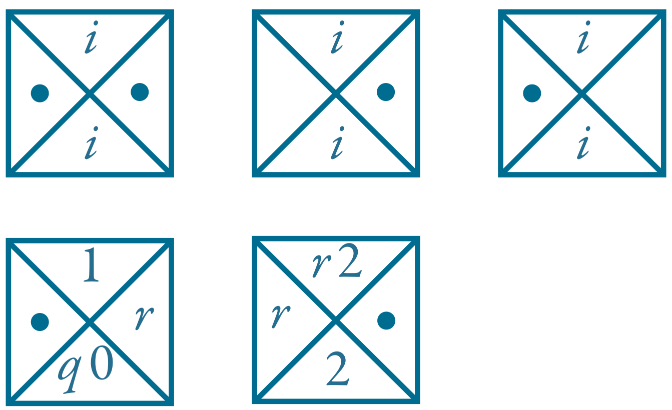

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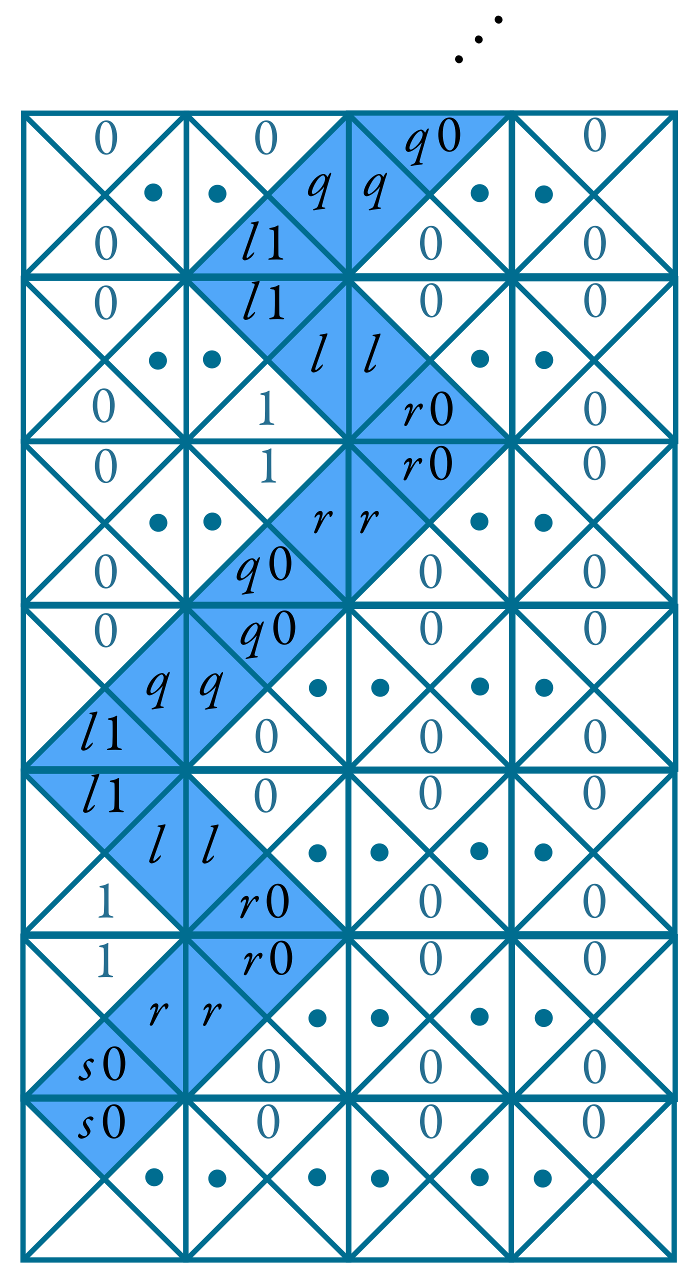

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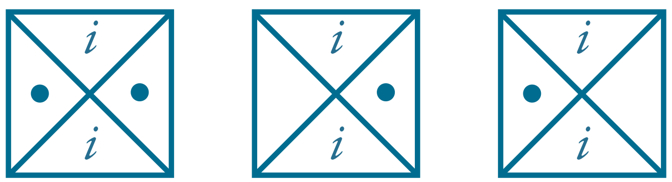

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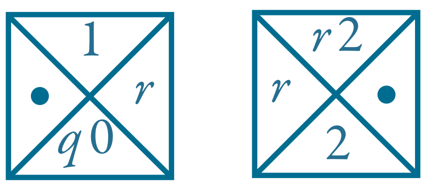

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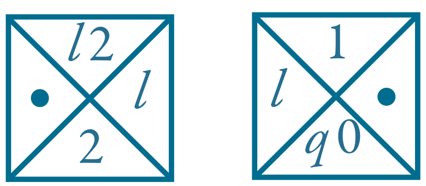

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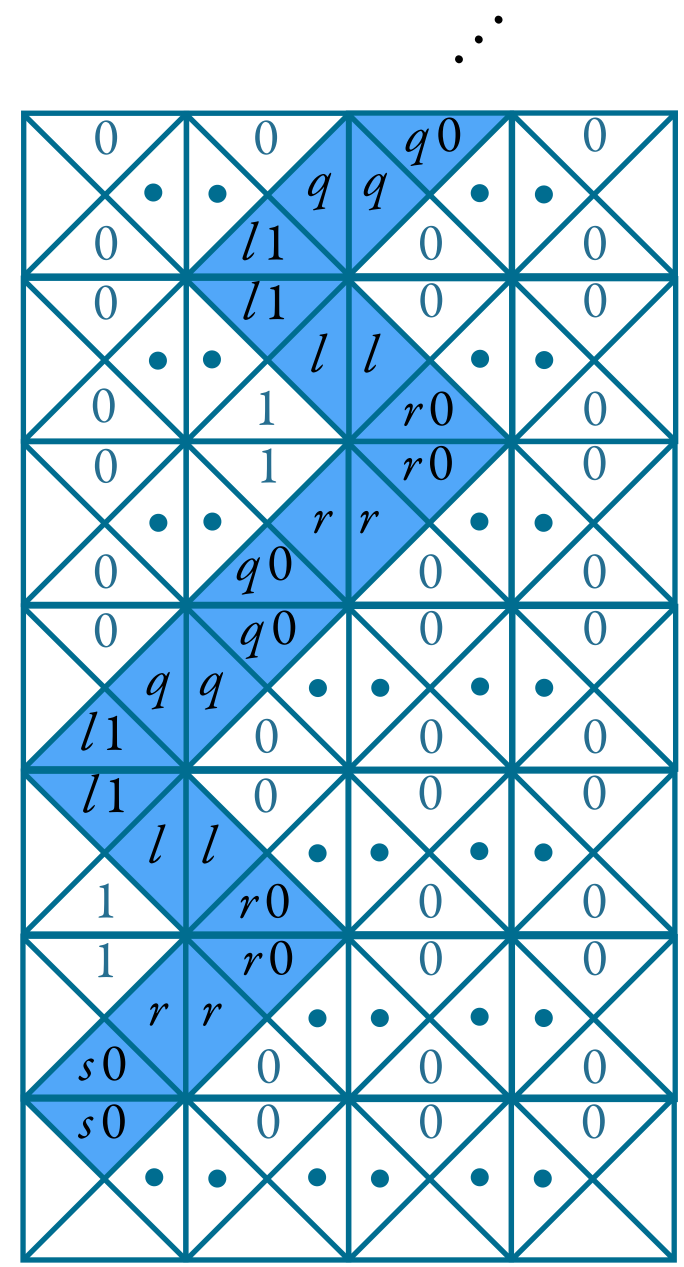

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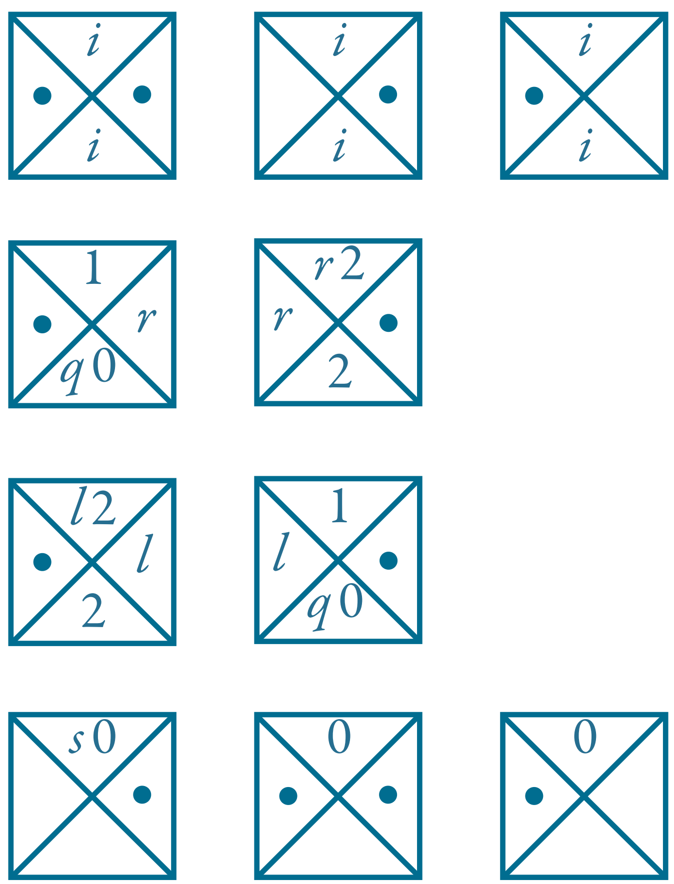

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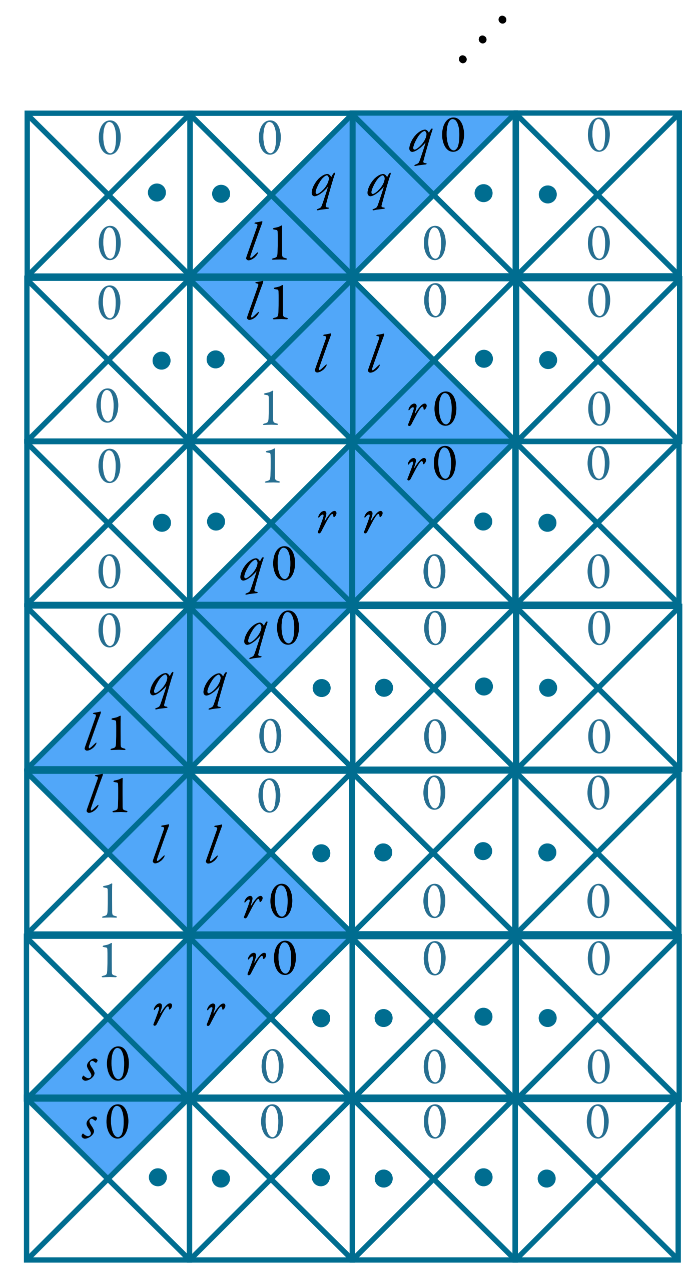

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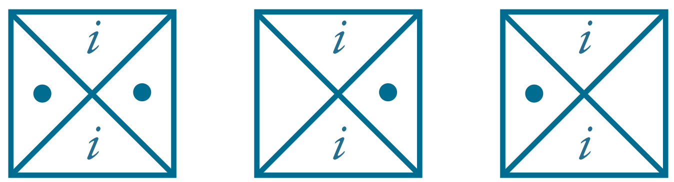

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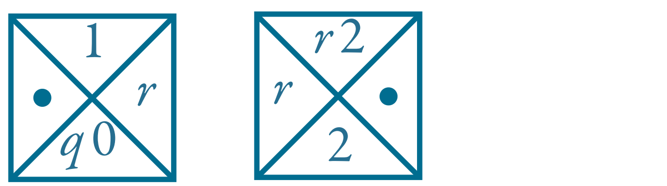

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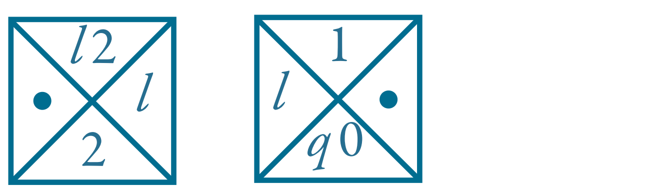

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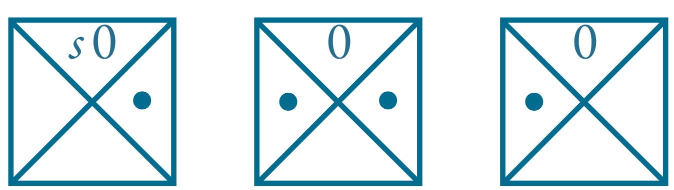

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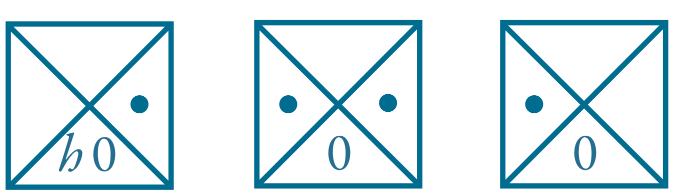

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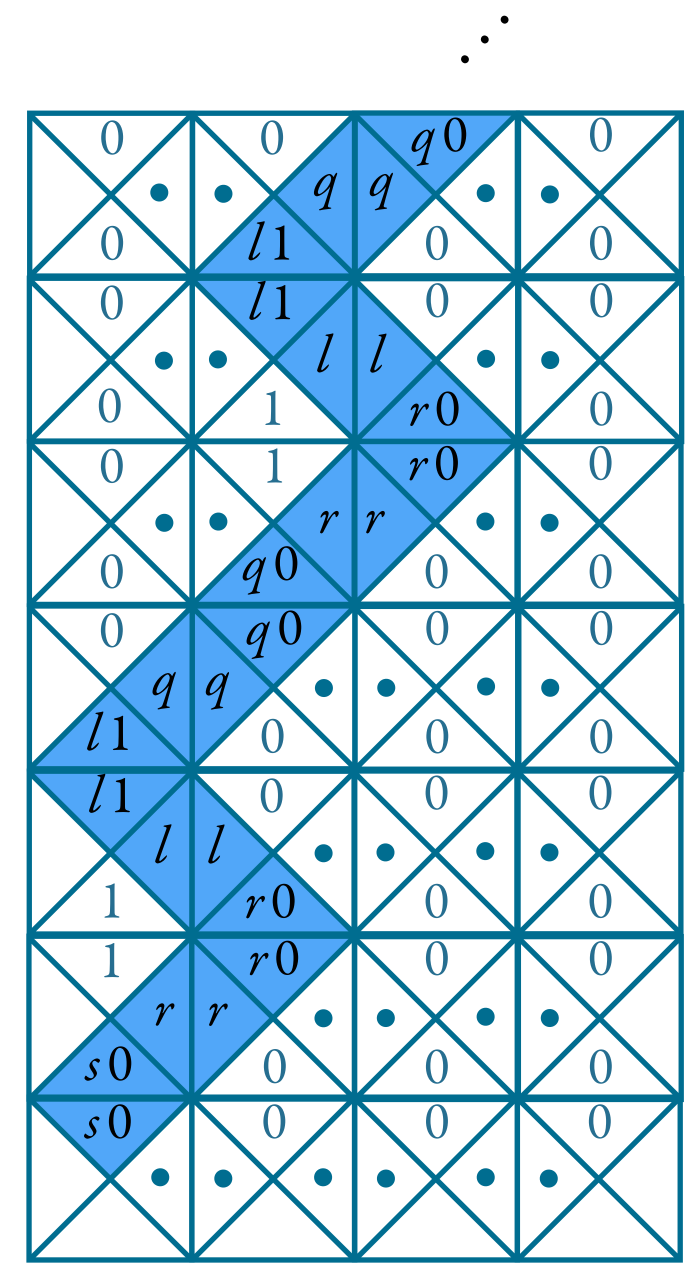
. . .



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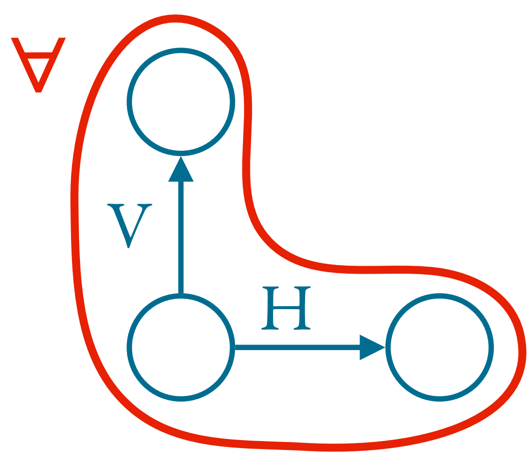

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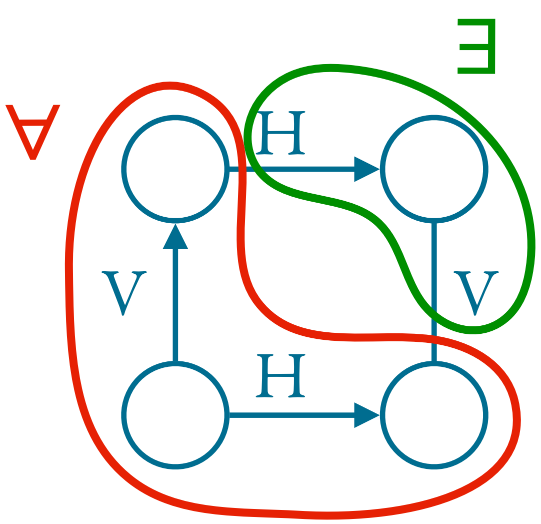

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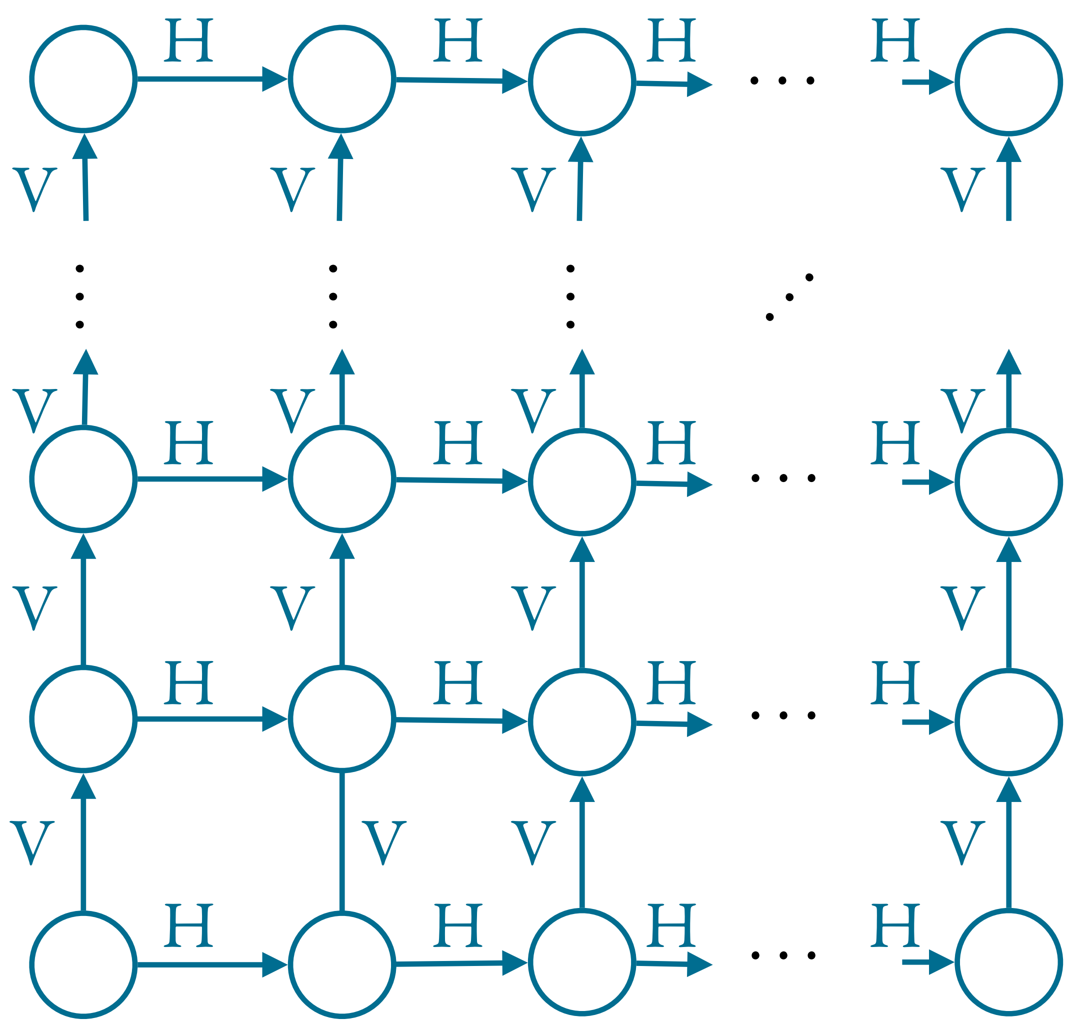

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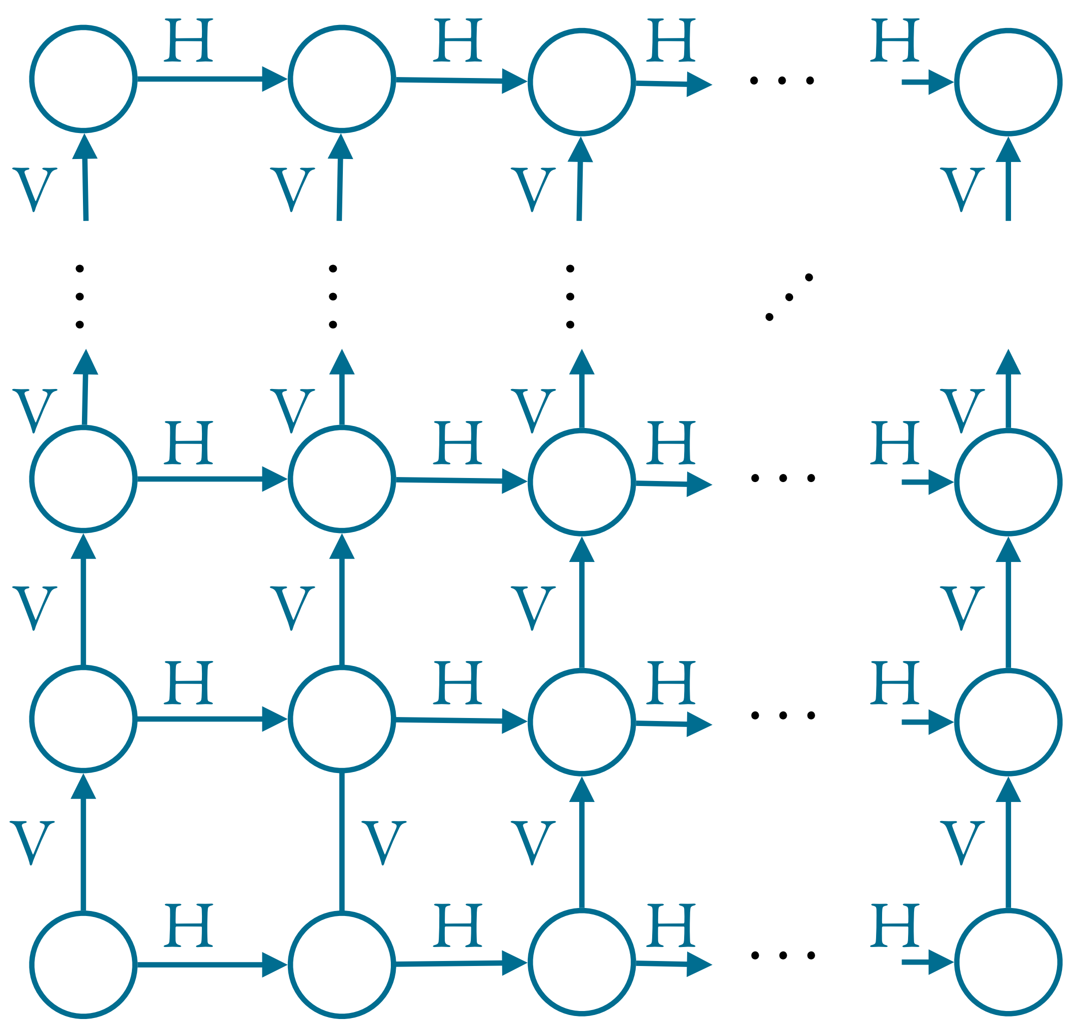

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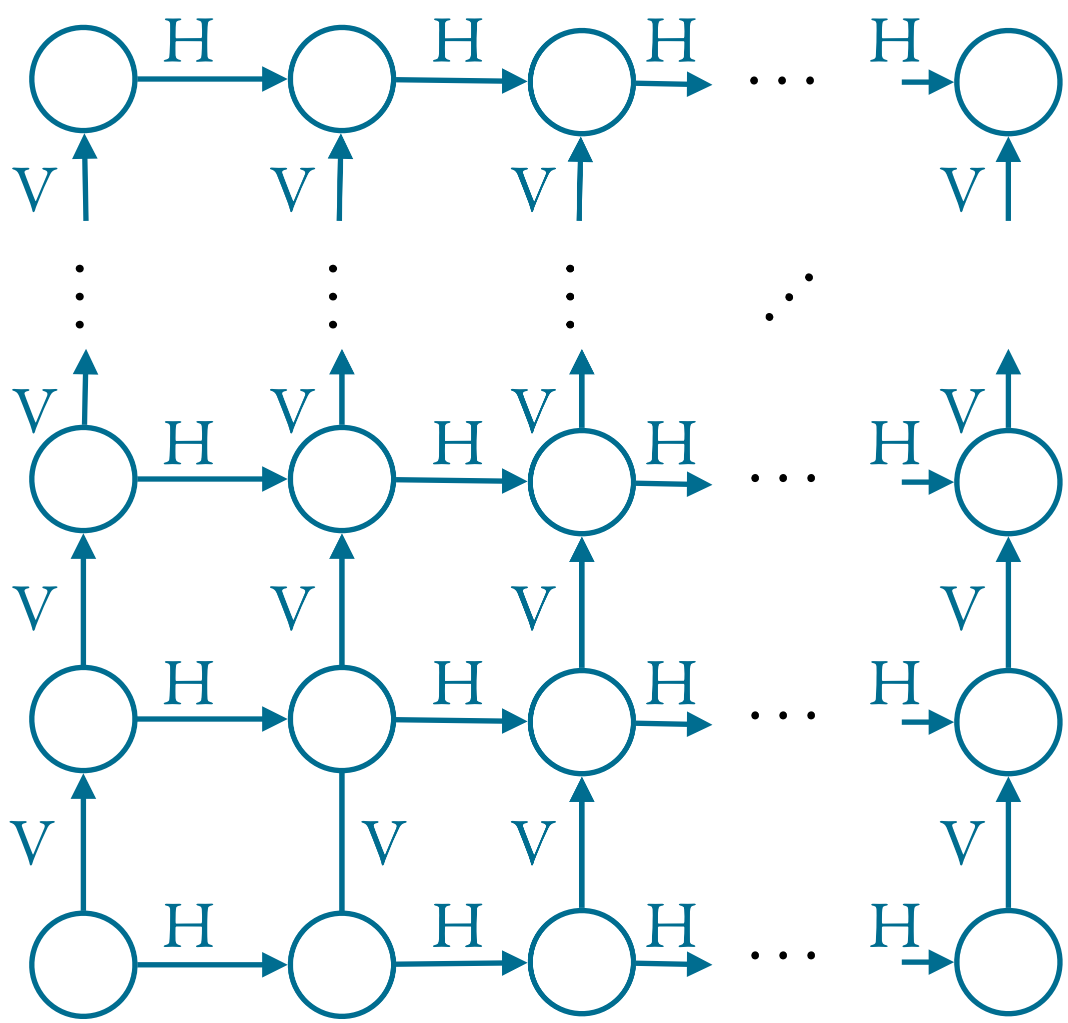

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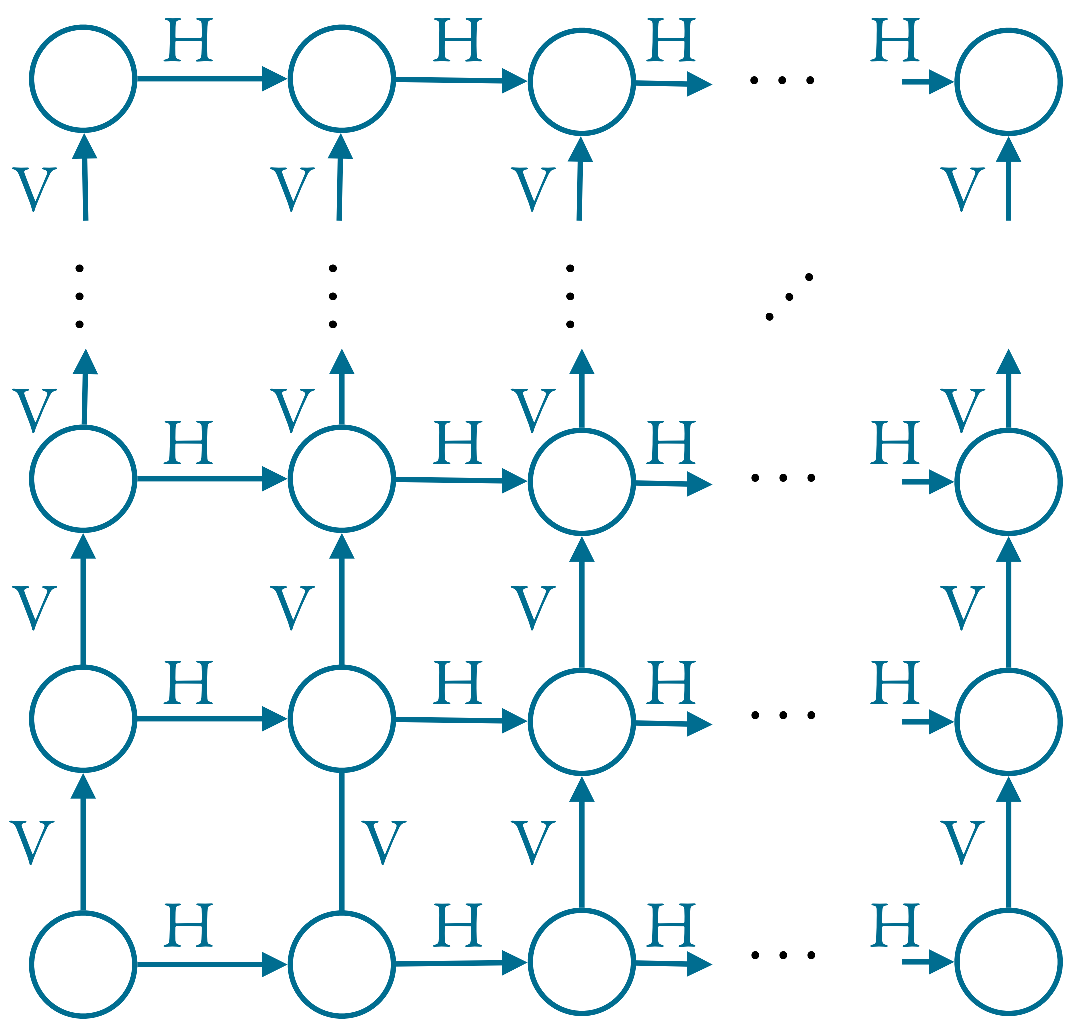

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4. **Borders are white.**

Recap + quiz

- Model-checking for FO (does $M \vDash \phi$?) is **PSPACE**-complete
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- Validity for FO? (Problem def.: does $M \vDash \phi$ for every M?)
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Can you recall the complexity of analogous problems for

- Propositional logic?
- QBF?

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Example

$FO[\mathbb{N},<] = \{\ \exists x\,(x=x),\ \forall x \exists y\ x<y,\ \exists y\, \forall x\ \neg(x<y),\ \forall x \forall y\ x=y \vee x<y \vee y<x,\ \ldots\ \}$

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(notation abuse: relation = is often present, but not explicitly listed
any symbol R is often identified with its relation R^M)

Some fancy FO theories

(Skip)

$$\begin{aligned}
\mathrm{FO}[\mathbb{N}, +, \cdot\,] &= \text{Peano arithmetic} \\
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Logical reductions

Reduction from P to P':

Algorithm A that solves P by using an oracle that returns solutions to P'

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FO interpretation of M into M': a mapping $\alpha : R \mapsto \alpha_R$ such that

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$$\approx (\mathbb{Q}, \leq)$$

$$\alpha_{\leq_{\text{inorder}}}(x, y) = \exists x', y', z \;\; (x=z\cdot 0\cdot x' \wedge y=z\cdot 1\cdot y') \vee (x=y\cdot 0\cdot x') \vee (y=x\cdot 1\cdot x')$$

Logical reductions

In fact, an FO interpretation of M into M' is more complex (and powerful)

- <u>definitions of relations</u>: $\alpha_R(\bar{x})$ such that $R^M = \{\, \bar{u} \mid M'[\bar{x} := \bar{u}] \vDash \alpha_R(\bar{x}) \,\}$

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- <u>quotient</u>: $\alpha_=(\bar{x},\bar{y})$ such that $M[\ldots] \vDash (\bar{x}=\bar{y})$ iff $M'[\ldots] \vDash \alpha_=(\bar{x},\bar{y})$

  (e.g. to interpret $(\mathbb{Q},+,\cdot)$ into $(\mathbb{Z},+,\cdot)$)

Logical reductions

Given M' and an FO interpretation $\alpha = (\alpha_U, \alpha_=, \alpha_R, \alpha_S, \ldots)$
the interpreted model is $\alpha(M') = (U^M, R^M, S^M, \ldots)$ where

- $U^M = \{\, [\bar{u}]_\approx \mid M'[\bar{x} := \bar{u}] \vDash \alpha_U(\bar{x}) \,\}$

- $\bar{u} \approx \bar{v}$ iff $M'[\bar{x} := \bar{u}, \bar{y} := \bar{v}] \vDash \alpha_=(\bar{x},\bar{y})$

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Theorem If $\alpha = (\alpha_U, \alpha_=, \alpha_R, \alpha_S, \dots)$ is an FO interpretation of M into M' then FO[M] *reduces to* FO[M'], namely, there is an algorithm A_α

for all ϕ $\quad M \vDash \phi$ iff $M' \vDash A_\alpha(\phi)$

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$\mathrm{FO}[\mathbb{N}, +, \cdot\,]$	$=$	Peano arithmetic
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Proof by reduction from undecidable Hilbert's 10th problem... [Matiyasevic '70]

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Given a polynomial $p(x,y,z,\ldots)$
tell whether $p(x,y,z,\ldots) = 0$ for *some integers* x, y, z

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1. Given polynomial $p(x,y,z,\ldots)$, inductively construct $\phi_p(x,y,z,\ldots,t)$ such that
$(\mathbb{Z},+,\cdot, x,y,z,\ldots,t) \models \phi_p$ iff $p(x,y,z)=t$
2. Interpret $(\mathbb{Z},+,\cdot,0)$ into $(\mathbb{N},+,\cdot)$

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Algebraic geometry

Programs verification

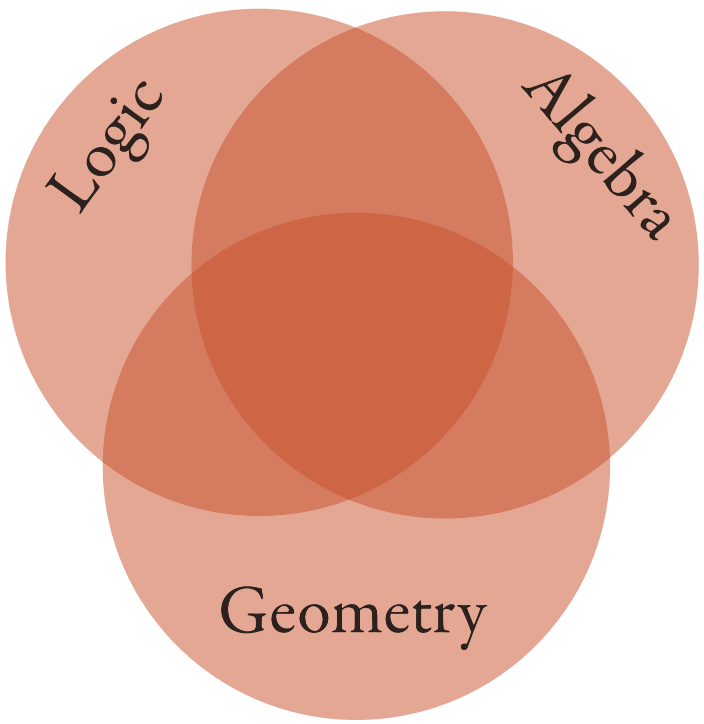
Continuous & discrete dynamical systems

Computer graphics

Robotics

Coding theory & Cryptography

Grammars & Transducers



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temporarily assume formulas are over the reals or the rationals...

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Example

temporarily assume formulas are over the reals or the rationals...

$$\exists z\ \alpha(x,y,z) = \exists z\ (2x + 4y - 3z \leq 7) \wedge (3x - y + 2z \leq -4)$$

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Theorem [Presburger '29] Every FO formula ϕ over $(Z,+,0,1,\leq,|)$ can be effectively transformed into an <u>equivalent quantifier-free</u> formula ϕ^*

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FO[{0,1}, =] — The FO theory of Boolean algebra

Lemma Given any QBF ϕ without free variables,
one can construct an FO formula ϕ^* such that

$$\models \phi \quad \text{iff} \quad (\{0,1\}, =) \models \phi^*$$

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Corollary FO[{0,1}, =] encodes the set of valid QBF formulas

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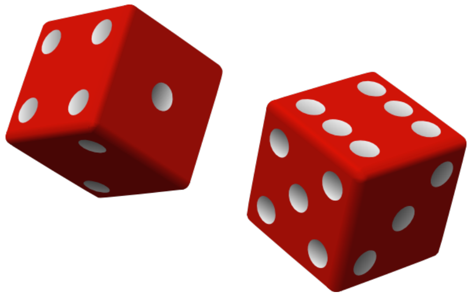
FO[V_R, E_R] — The FO theory of the "random" graph

A different perspective and a coarser view on expressiveness…

What percentage of finite graphs verify a given FO sentence?



Probability of a formula

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Example For ϕ = "the graph is complete",

we have $P_n[\phi] = \dfrac{1}{2^{n(n-1)}}$

and hence $P_\infty[\phi] = 0$

Probability of a formula

Theorem (0/1 Law)
[Glebskii et al. '69, Fagin '76]

Every FO formula ϕ is
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$P_\infty[\phi] = 1/2$

$P_\infty[\phi]$ not even defined

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- ϕ = "even number of edges" — $P_\infty[\phi] = 1/2$
- ϕ = "even number of nodes" — $P_\infty[\phi]$ not even defined
- ϕ = "more edges than nodes" — $P_\infty[\phi] = 1$
  (yet not FO-definable...)

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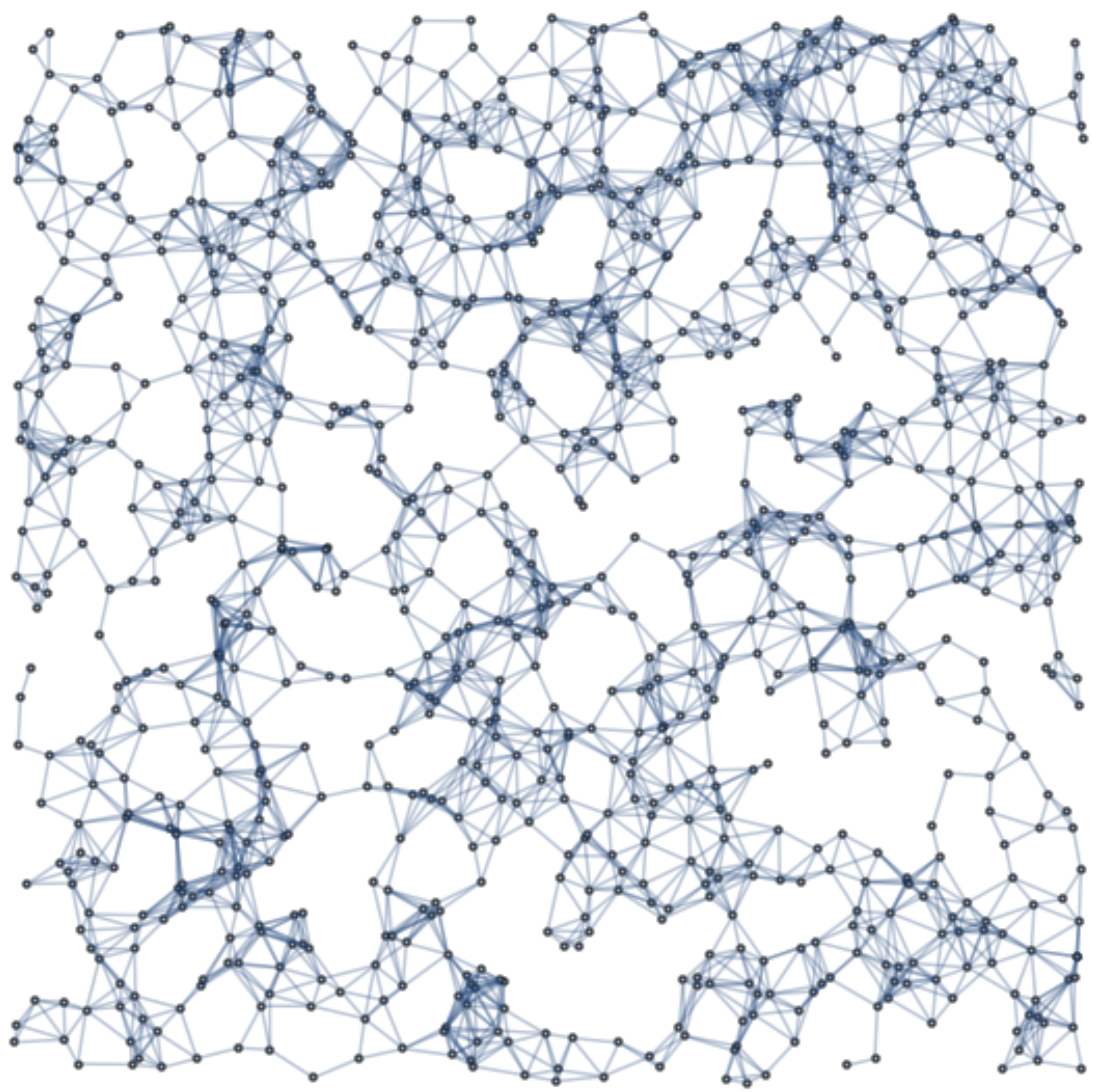
The “random” infinite graph

Every FO formula ϕ is either almost surely true or almost surely false, *and this depends on whether* $(V_R, E_R) \vDash \phi$

The “random” graph (V_R, E_R)



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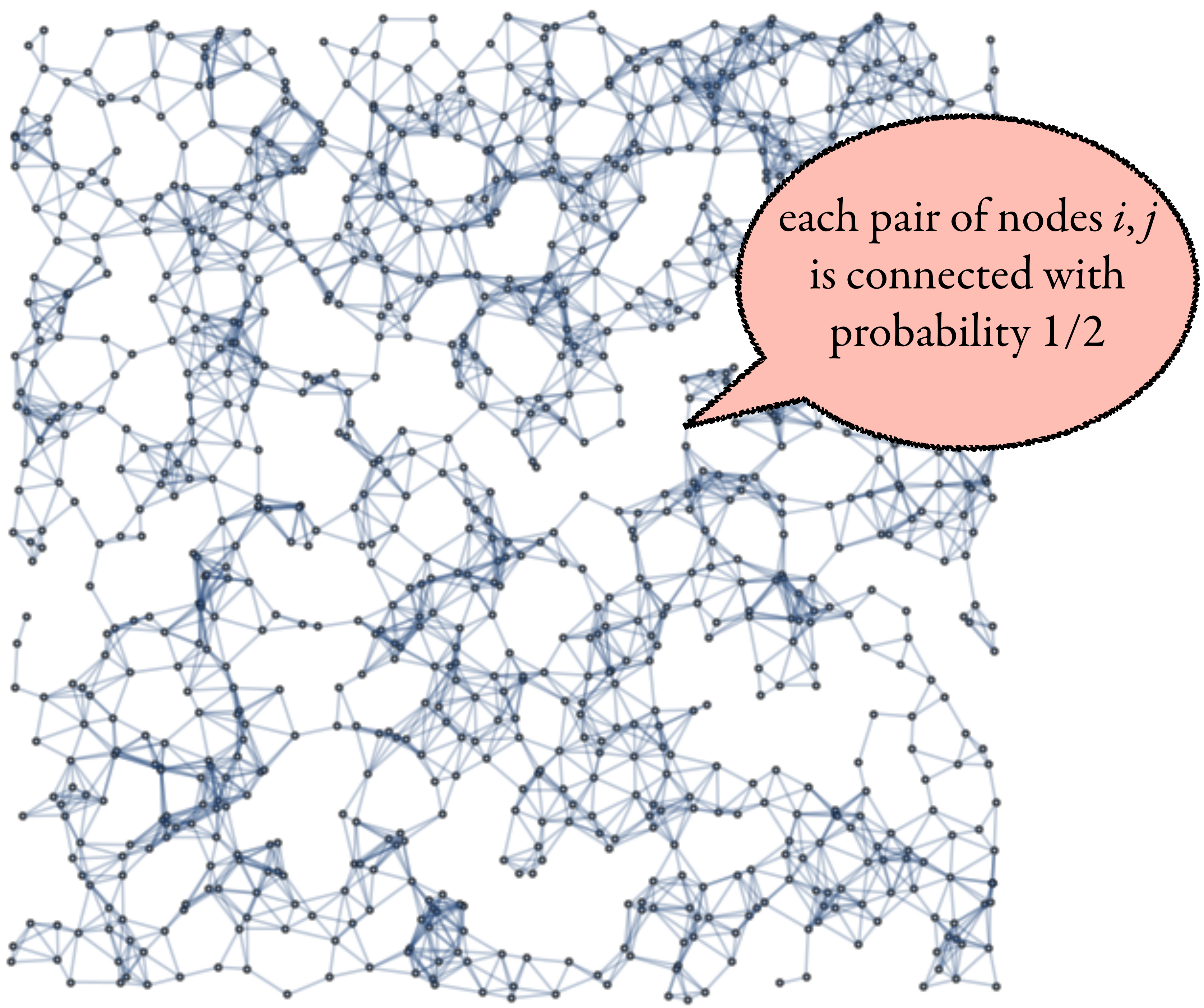

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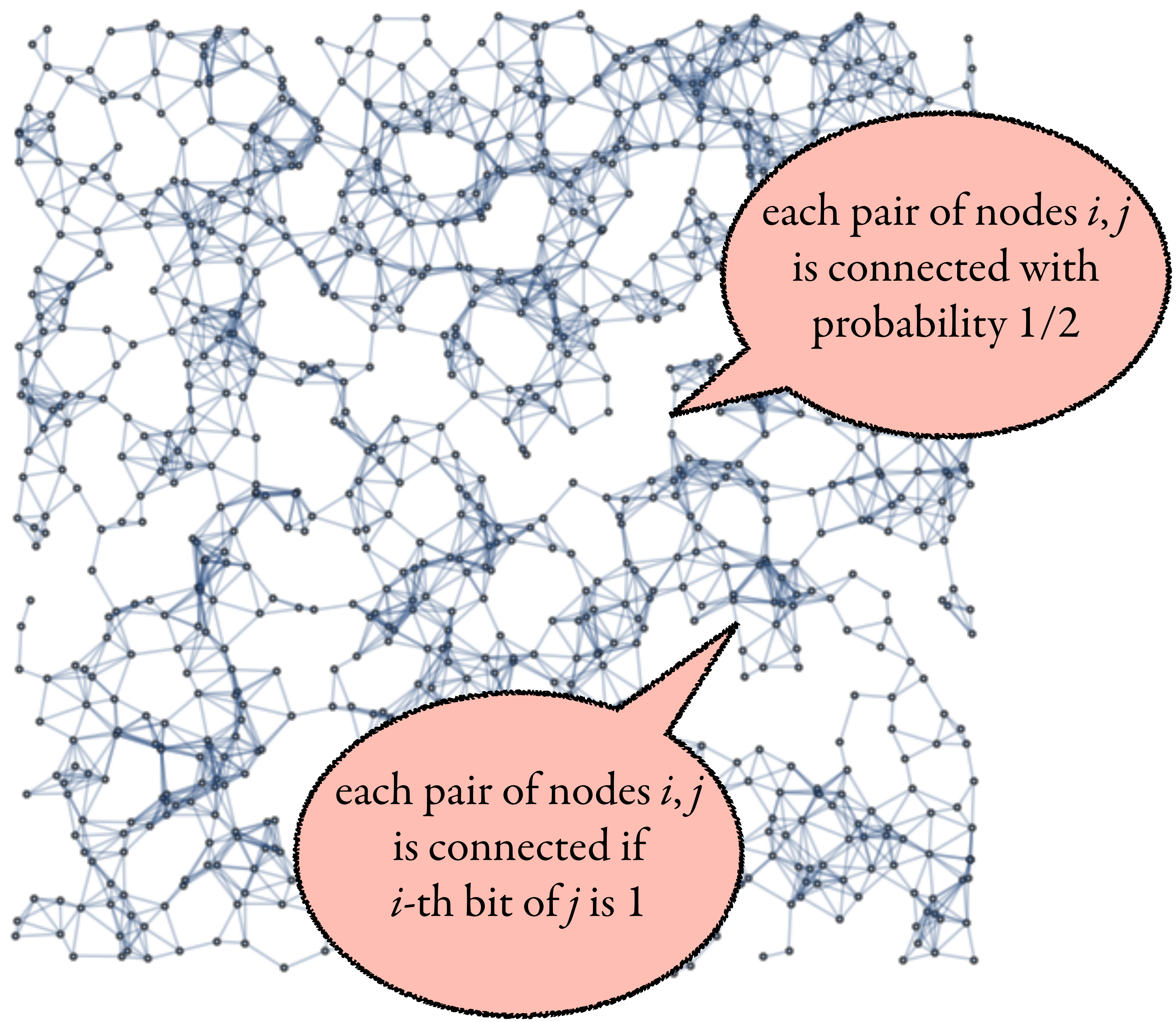

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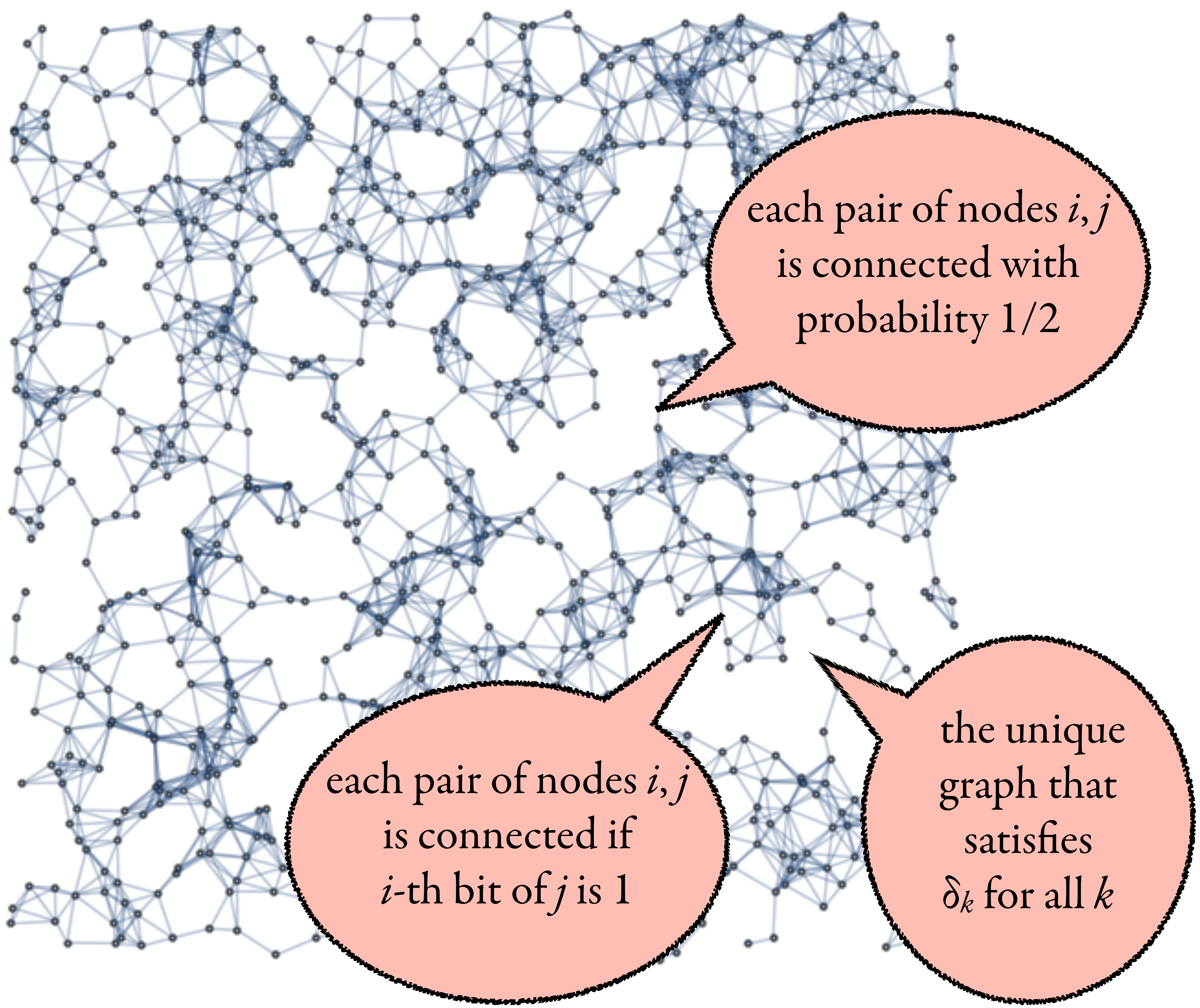

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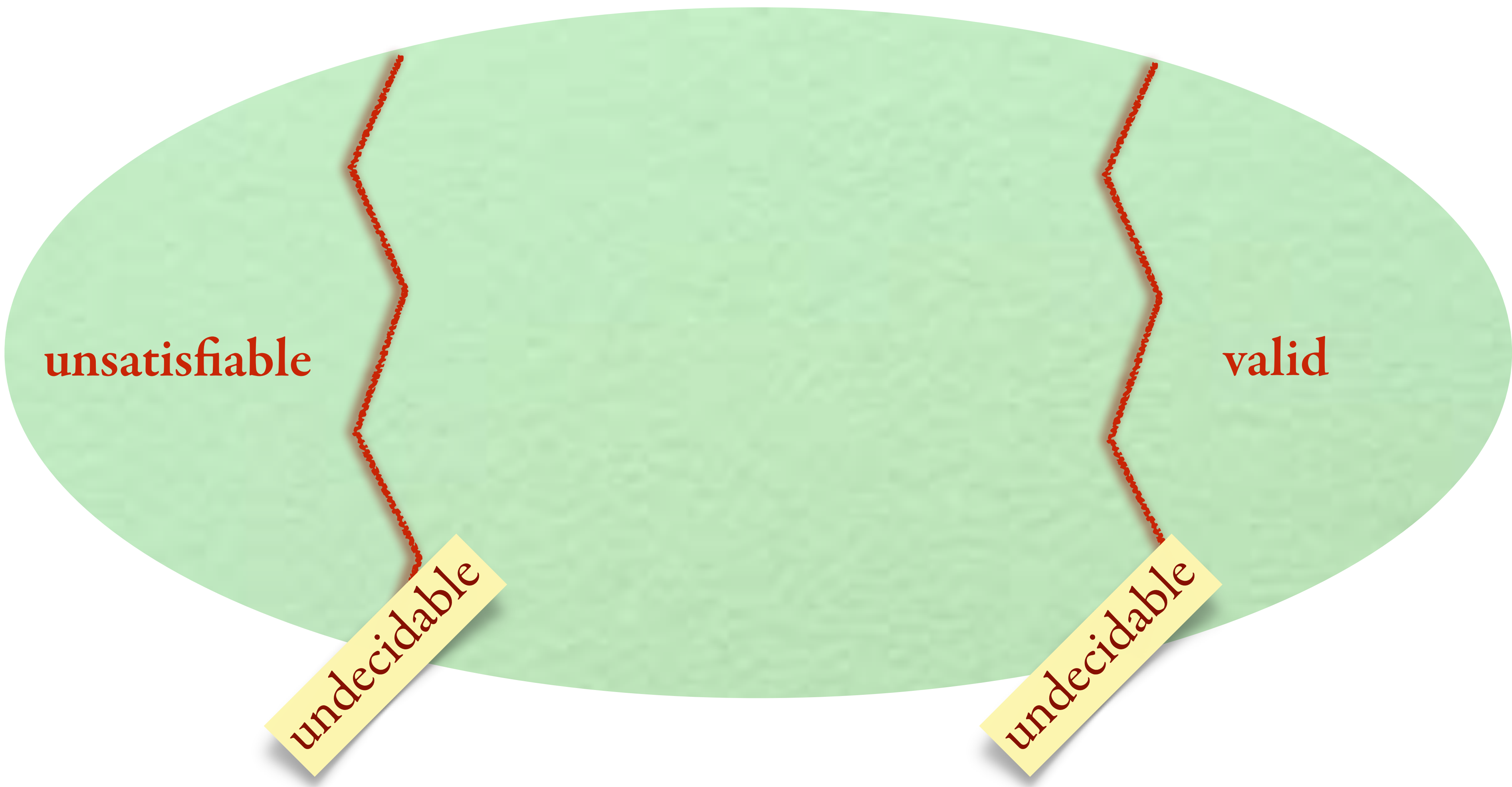

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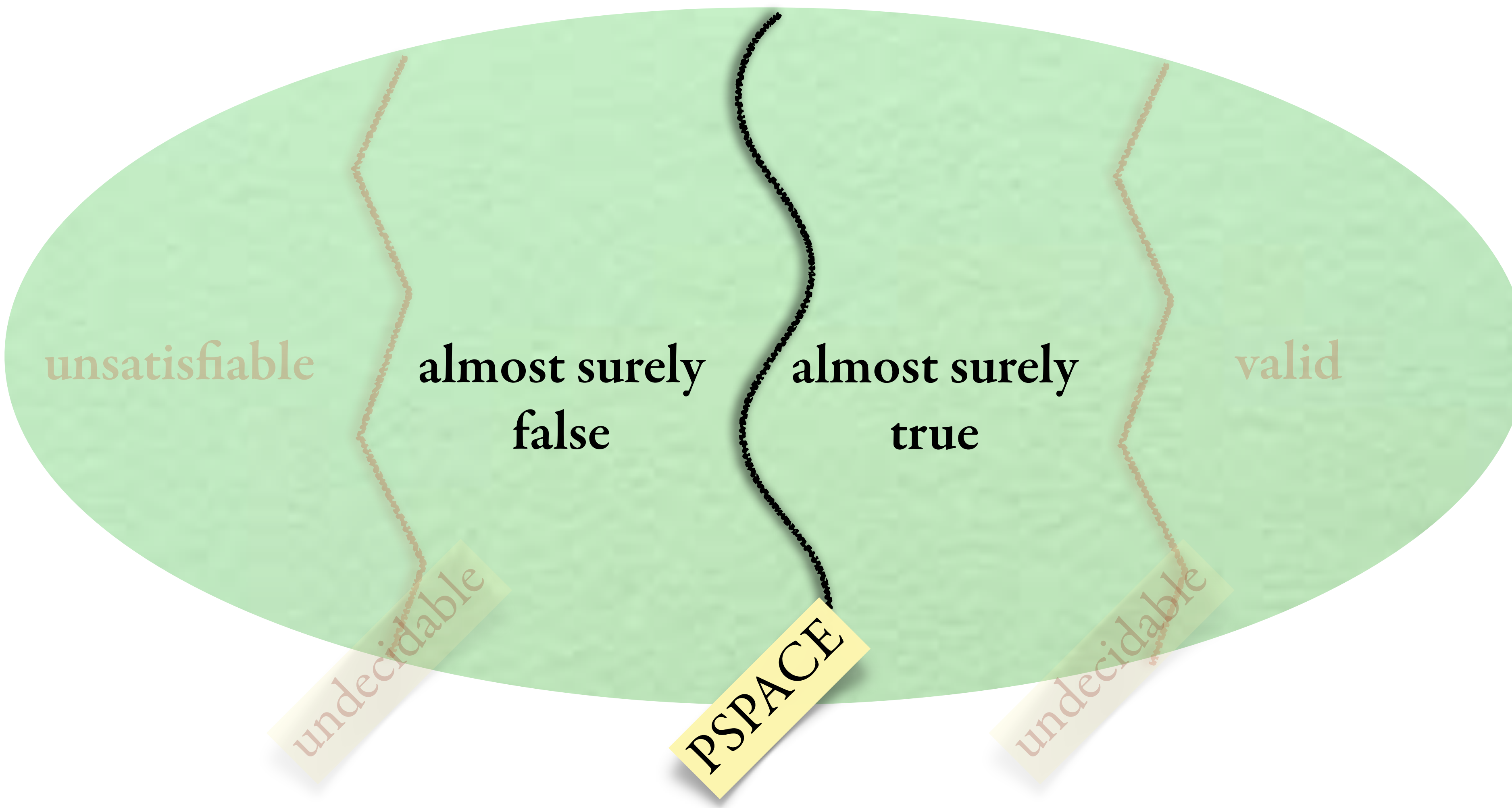

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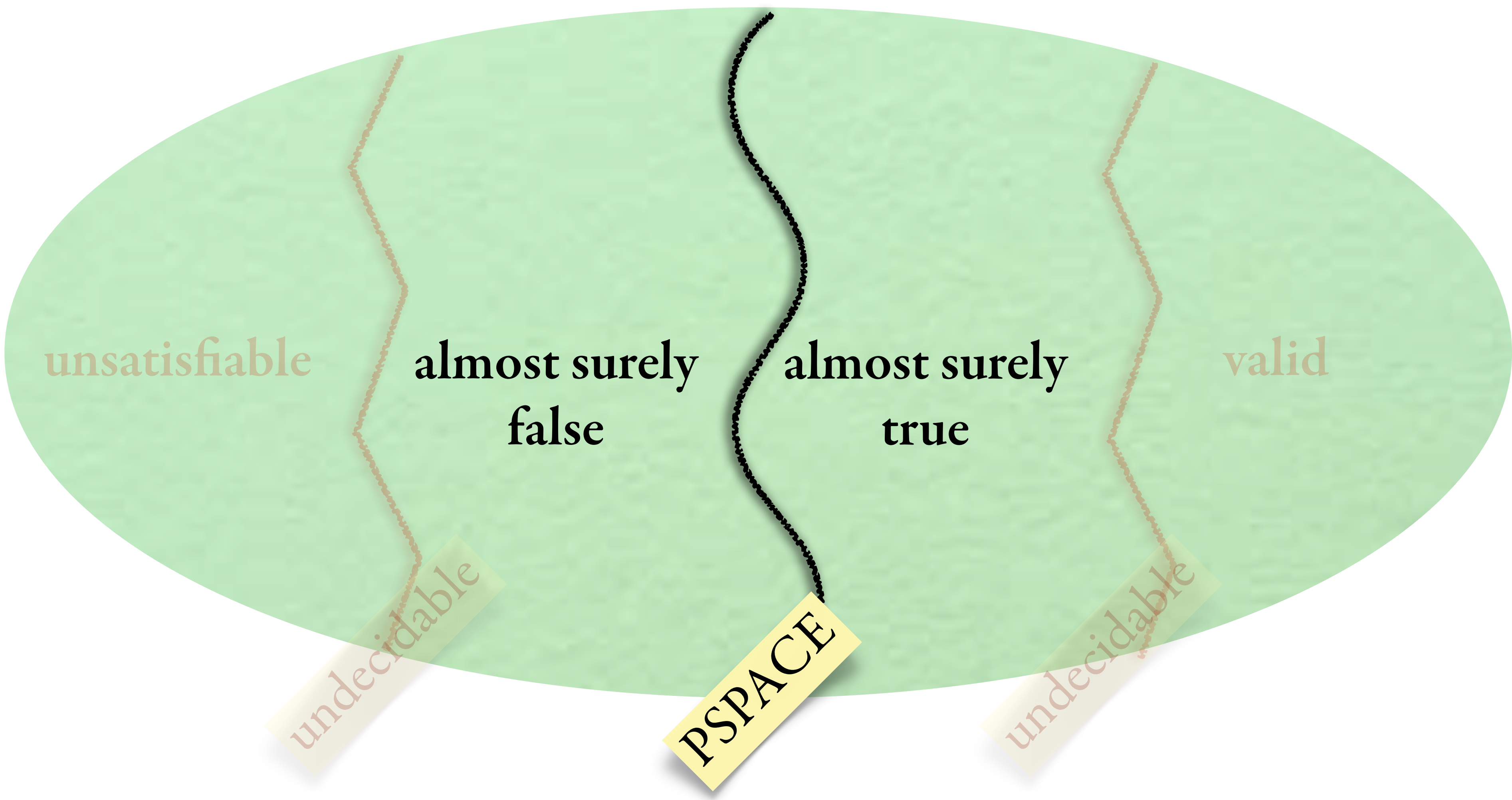

Model-checking on large graphs/databases

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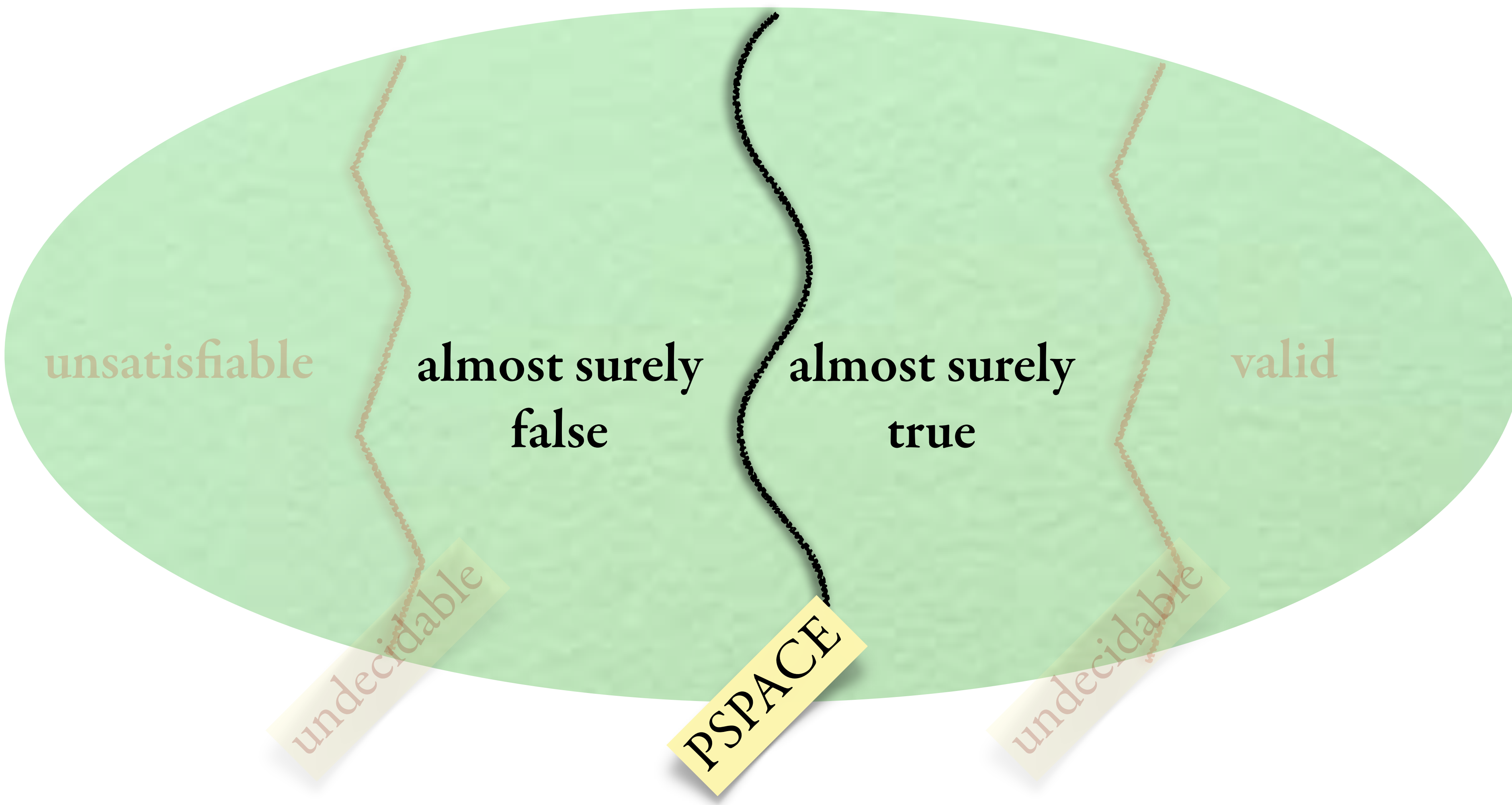

Disclaimer:

0/1 Law only applies applies to unconstrained graphs

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Things to remember

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- FO is cool and quite expressive

- Model-checking is decidable (in **PSPACE**) when the universe is finite
  Satisfiability, validity, equivalence are all undecidable (reduction from Domino)

- For infinite universes, one can fix a model and study its FO theory
  Some FO theories are decidable, some are not

- Some FO theories can be reduced to others via FO interpretations